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A Bound for the Spectral Radius of a Matrix

by

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It is the purpose of this note to derive a bound for the eigenvalues of a complex matrix $A = (a_{ij})$ in terms of a_{ij}. The estimate will be derived by constructing an integral equation with degenerate kernel for which each finite eigenvalue is the reciprocal of a non-zero eigenvalue of A. The precise result is:

Theorem: Let $A = (a_{ij})$ be an $n \times n$ complex matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A (not necessarily distinct) but one of which is $\neq 0$, then

$$(1) \qquad |\lambda|^2_{max} = \max |\lambda_i|^2$$

$$\leq G^2(a_{ij}) \equiv \sum_{\ell=1}^{n} (2\ell-1)^{-1}\{\alpha_{\ell\ell}+\beta_{\ell\ell}\} + \sum_{\substack{\ell,k=1\\ \ell\neq k}}^{n} (\ell+k-1)^{-1}\{\alpha_{\ell k}+\beta_{\ell k}\}$$

with

$$(2) \qquad \alpha_{\ell k} = \sum_{j=1}^{n} (2j-1)^{-1} q_{j\ell}\bar{q}_{jk}, \quad \beta_{\ell k} = \sum_{\substack{j,m=1\\ j\neq m}}^{n} (j+m-1)^{-1} q_{j\ell}\bar{q}_{mk}$$

* The author wishes to greatfully acknowledge support for this research by the National Aeronautics Space Administration, NASA Grant NGR 05-010-008.

where

$$(3) \qquad q_{k\ell} = \sum_{m=1}^{n} a_{\ell m} b_{mk}, \qquad 1 \le j,\ell \le n,$$

with

$$(4) \qquad b_{mk} = (k+m-1) \prod_{\substack{p=1 \\ p\neq m}}^{n} \frac{p+k-1}{p-m} \prod_{\substack{r=1 \\ r\neq k}}^{n} \frac{r+m-1}{r-k} .$$

Remark: (b_{mk}) is a universal matrix in the sense it is independent of A.

Before we prove the theorem we shall prove the following.

Lemma: *Consider the integral equation*

$$(5) \qquad \varphi(s) = \mu \int_0^1 K(s,t)\, \varphi(t)dt$$

where

$$(6) \qquad K(s,t) = \sum_{\ell=1}^{n} s^{\ell-1} q_\ell(t)$$

with

$$(7) \qquad q_\ell(t) = \sum_{j=1}^{n} q_{j\ell} t^{j-1}$$

and $q_{j\ell}$ *is given by* (3). *If* $\lambda \neq 0$ *is an eigenvalue of* A, *then* $\mu = \lambda^{-1}$ *is an eigenvalue of* (5).

<u>Proof of Lemma</u>: First we note that

$$\int_0^1 t^{j-1} q_\ell(t)\,dt = a_{\ell j}, \qquad 1 \le j,\ell \le n. \tag{8}$$

This follows since

$$\begin{aligned}
\int_0^1 t^{j-1} q_\ell(t)dt &= \sum_{k=1}^{n} q_{k\ell} \int_0^1 t^{j+k-2} dt && \text{by (7)} \\
&= \sum_{k=1}^{n} (j+k-1)^{-1} q_{k\ell} \\
&= \sum_{k=1}^{n} (j+k-1)^{-1} \sum_{m=1}^{n} a_{\ell m} b_{mk} && \text{by (3)} \\
&= \sum_{k=1}^{n} (j+k-1)^{-1} \sum_{m=1}^{n} (k+m-1) \prod_{\substack{p=1\\p\neq m}}^{n} \frac{p+k-1}{p-m} \prod_{\substack{r=1\\r\neq k}}^{n} \frac{r+m-1}{r-k}\, a_{\ell m} \\
&= \sum_{m=1}^{n} a_{\ell m} \left\{ \sum_{k=1}^{n} (j+k-1)^{-1}(k+m-1) \prod_{\substack{p=1\\p\neq m}}^{n} \frac{p+k-1}{p-m} \prod_{\substack{r=1\\r\neq k}}^{n} \frac{r+m-1}{r-k} \right\}.
\end{aligned} \tag{9}$$

<u>Case</u> 1 $m = j$. The j^{th} term of the sum in m becomes:

$$a_{\ell j} \sum_{k=1}^{n} \prod_{\substack{p=1\\p\neq j}}^{n} \frac{p+k-1}{p-j} \prod_{\substack{r=1\\r\neq k}}^{n} \frac{r+j-1}{r-k}\ .$$

If we let

$$\psi_{n,j}(x) = \prod_{\substack{p=1 \\ p\neq j}}^{n} \frac{p-x}{p-j}, \qquad \varphi_{n,k}(x) = \prod_{\substack{r=1 \\ r\neq k}}^{n} \frac{r-x}{r-k},$$

then the above becomes

$$a_{\ell j} \sum_{k=1}^{n} \psi_{n,j}[-(k-1)]\ \varphi_{n,k}[-(j-1)]$$

in which $\psi_{n,j}(x)$ is a polynomial of degree $\le n-1$ in x and thus

$$\psi_{n,j}[-(k-1)] = P_{n,j}(k)$$

is a polynomial of degree $\le n$ in k. By the Lagrange identities

$$\sum_{j=1}^{n} j^m \varphi_{n,j}(x) = x^m, \qquad m = 0,1,\ldots,n.$$

Thus:

$$\text{(10)} \qquad \sum_{k=1}^{n} \psi_{n,j}[-(k-1)]\ \varphi_{n,k}[-(j-1)] = \sum_{k=1}^{n} P_{n,j}(k)\ \varphi_{n,k}[-(j-1)]$$

$$= P_{n,j}[-(j-1)]$$

$$= \psi_{n,j}(j)$$

$$= 1\ .$$

<u>Case</u> 2 $m \neq j$. For any term, not j, of the sum in m we get:

$$\sum_{k=1}^{n} a_{\ell m}(j+k-1)^{-1}(k+m-1)\ \psi_{n,m}[-(k-1)]\ \varphi_{n,k}[-(m-1)]. \tag{11}$$

Note that

$$(j+k-1)^{-1}\ \psi_{n,m}[-(k-1)] = (j+k-1)^{-1} \prod_{\substack{p=1 \\ p\neq m}} \frac{p+k-1}{p-m}$$

is a polynomial in k of degree $\leq n-2$ since $m \neq j$ and thus $p = j$ occurs in the product. As in Case 1, since

$$(j+k-1)^{-1}(k+m-1)\ \psi_{n,m}[-(k-1)]$$

is a polynomial of degree $\leq n-1$ in k, we get, using the Lagrange identities that (11) becomes:

$$a_{\ell m}(j-m)^{-1}(m-m)\ \psi_{n,m}[m] = 0. \tag{12}$$

Thus combining the results for Case 1 and Case 2 above we get (8).

To complete the proof of the lemma, let $x = (x_1,x_2,\ldots,x_n)^T$ be an eigenvector of A corresponding to the eigenvalue λ, i.e.

$$Ax = \lambda x. \tag{13}$$

Let

$$\varphi(s) = \sum_{j=1}^{n} x_j s^{j-1}, \tag{14}$$

then $\varphi(s)$ is $\not\equiv 0$ and is a solution of

$$\varphi(s) = \lambda^{-1} \int_0^1 K(s,t)\, \varphi(t)dt.$$

To see this note that by (6)

$$\begin{aligned}
\lambda^{-1} \int_0^1 K(s,t)\, \varphi(t)dt &= \lambda^{-1} \int_0^1 K(s,t) \sum_{j=1}^{n} x_j t^{j-1} dt \\
&= \lambda^{-1} \sum_{\ell=1}^{n} s^{\ell-1} \sum_{j=1}^{n} x_j \int_0^1 q_\ell(t) t^{j-1} dt \\
&= \lambda^{-1} \sum_{\ell=1}^{n} s^{\ell-1} \sum_{j=1}^{n} a_{\ell j} x_j \qquad \text{by (8)} \\
&= \lambda^{-1} \sum_{\ell=1}^{n} s^{\ell-1} \lambda x_\ell \qquad \text{by (13)} \\
&= \varphi(s) \qquad \text{by (14).}
\end{aligned}$$

This completes the proof of the lemma.

<u>Proof of theorem:</u> As is well known from the theory of integral equations all eigenvalues of

$$\varphi(s) = \mu \int_0^1 K(s,t)\, \varphi(t)dt \tag{15}$$

lie outside the disc

$$|\mu|\, \|K\|_2 < 1$$

where

$$\|K\|_2^2 = \int_0^1 \int_0^1 |K(s,t)|^2 \, ds \, dt,$$

i.e., if μ is an eigenvalue of (15) then

$$|\mu|^{-1} \leq \|K\|_2.$$

Thus in particular for

$$|\lambda|_{max} = \max|\lambda_i| \neq 0$$

of (1) we have

$$|\lambda|_{max} \leq \|K\|_2.$$

In order to calculate $\|K\|_2$ note that

$$(16) \qquad \sum_{\ell=1}^{n} s^{\ell-1} q_\ell(t) \sum_{k=1}^{n} s^{k-1} \overline{q_k(t)} = \sum_{\ell=1}^{n} s^{2\ell-2} q_\ell(t) \overline{q_\ell(t)} + \sum_{\substack{\ell,k=1 \\ \ell \neq k}} s^{\ell+k-2} q_\ell(t) \bar{q}_k(t).$$

But by (7),

$$(17) \qquad q_\ell(t) \overline{q_\ell(t)} = \sum_{j=1}^{n} q_{j\ell} t^{j-1} \sum_{m=1}^{n} \overline{q_{m\ell}} \, t^{m-1}$$

$$= \sum_{j=1}^{n} q_{j\ell} \bar{q}_{j\ell} t^{2j-2} + \sum_{\substack{j,m=1 \\ j \neq m}}^{n} t^{j+m-2} q_{j\ell} \bar{q}_{m\ell},$$

and for $\ell \neq k$

$$(18)\qquad q_\ell(t)\,\overline{q_k(t)} = \sum_{j=1}^{n} q_{j\ell} t^{j-1} \sum_{m=1}^{n} \bar{q}_{mk} t^{m-1}$$

$$= \sum_{j=1}^{n} q_{j\ell}\,\bar{q}_{jk} t^{2j-2} + \sum_{\substack{j,m=1\\ j\neq m}}^{n} t^{j+m-2} q_{j\ell}\bar{q}_{mk}.$$

Thus combining (16), (17) and (18) gives

$$\|K\|_2^2 = \int_0^1 dt \int_0^1 ds\ K(s,t)\,\overline{K(s,t)} =$$

$$= \int_0^1 dt \left\{ \sum_{\ell=1}^{n} (2\ell-1)^{-1} q_\ell(t)\,\overline{q_\ell(t)} + \sum_{\substack{\ell,k=1\\ \ell\neq k}}^{n} (\ell+k-1)^{-1} q_\ell(t)\bar{q}_k(t) \right\}$$

$$= \sum_{\ell=1}^{n} (2\ell-1)^{-1} \left\{ \sum_{j=1}^{n} (2j-1)^{-1} q_{j\ell}\bar{q}_{j\ell} + \sum_{\substack{j,m=1\\ j\neq m}}^{n} (j+m-1)^{-1} q_{j\ell}\bar{q}_{m\ell} \right\}$$

$$+ \sum_{\substack{\ell,k=1\\ \ell\neq k}}^{n} (\ell+k-1)^{-1} \left\{ \sum_{j=1}^{n} (2j-1)^{-1} q_{j\ell}\bar{q}_{jk} + \sum_{\substack{j,m=1\\ j\neq m}}^{n} (j+m-1)^{-1} q_{j\ell}\bar{q}_{mk} \right\}$$

which concludes the proof of the theorem.

Example: We shall give an estimate, using the above theorem, for the matrix

$$A = \begin{bmatrix} 8 & 4 \\ 1 & \frac{2}{3} \end{bmatrix},$$

$$B = (b_{mk}) = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix},$$

$$Q = (q_{k\ell}) = (AB)^T = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\alpha = (\alpha_{\ell k}) = \begin{bmatrix} 64 & 0 \\ 0 & \frac{4}{3} \end{bmatrix},$$

$$\beta = (\beta_{\ell k}) = \begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix},$$

and

$$G^2(a_{ij}) = 64 + \frac{4}{9} + 8 = 72\tfrac{4}{9} \approx 72.444 \ .$$

Thus $\lambda_{max} = \max|\lambda_i| \leq G(a_{ij}) \approx 8.511$ where the actual eigenvalues are

$$\lambda = \frac{13 \pm \sqrt{157}}{3}, \qquad |\lambda|_{max} \approx 8.510 \ .$$

Comparing estimates with some of the familar bounds [1] for this matrix we see that

$$|\lambda|_{max} \leq \sum_{i,j=1}^{2} |a_{ij}| = 13\tfrac{2}{3}$$

or

$$|\lambda|_{max} \leq \Big(\sum_{1 \leq i,j \leq 2} |a_{ij}|^2 \Big)^{1/2} = 9.024$$

or

$$|\lambda|_{max} \leq \max_{1 \leq i,j \leq 2} |a_{ij}| \cdot 2 = 16 \ .$$

This shows, for this example, that the estimate of the theorem is the best of the above estimates.

Bibliography

[1] Taussky, O., and Marcus, M., Hermitian forms and eigenvalues, article in Survey of Numerical Analysis, edited by J. Todd, McGraw-Hill (1962), pp. 283.